Lecture 4

13.1 - Vector Valued Functions & Space Curves

A <u>vector-valued function</u> is a function whose output is a vector. We have already encountered one: the vector equation for a line

$$\vec{l}(t) = \vec{P}_0 + t\vec{v}$$

More generally, they will have the form

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$
$$= f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

The input variable (in this case it's t) is called the <u>parameter</u>. Since $\vec{r}$ is a function, we can ask about its domain. The domain of a vector-valued function is the "intersection" of the domains of its component functions, that is, the values common to the domains of each of f, g, and h.

<u>Ex</u>: What is the domain of $\vec{r}(t) = \langle \sqrt{4-t^2}, e^{-3t}, \ln(t+1) \rangle$?

Sol: First, we find the domains of each of the component functions:

function	$f(t)=\sqrt{4-t^2}$	$g(t)=e^{-3t}$	$h(t)=\ln(t+1)$
domain	$-2\le t\le 2$ $[-2,2]$	$-\infty<t<\infty$ $(-\infty,\infty)=\mathbb{R}$	$-1<t<\infty$ $(-1,\infty)$

The t-values in common to each of these are:

$$-1<t\le 2$$

So, the domain is $(-1,2]$.

◇

As with normal functions, we can take limits of vector-valued functions:

If $\vec{r}(t)=\langle f(t), g(t), h(t)\rangle$, then

$$\lim_{t\to a}\vec{r}(t)=\left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle$$

And this leads us to the definition of continuity for vector valued functions:

$\vec{r}(t)$ is <u>continuous at a</u> if

$$\lim_{t\to a}\vec{r}(t)=\vec{r}(a)$$

Let's look at some examples of vector-valued functions.

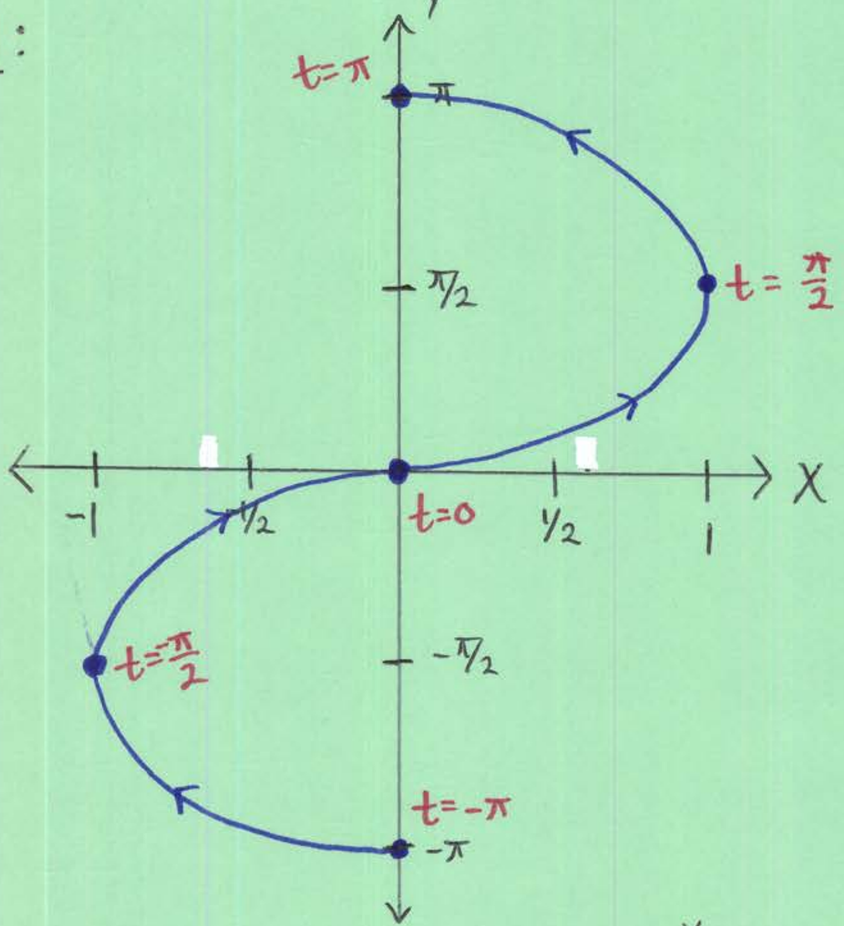
Ex: i) $\vec{r}(t) = \langle \sin t, t \rangle$, ii) $\vec{r}(t) = \langle 3\cos t, \sin t \rangle$

iii) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, iv) $\vec{r}(t) = \langle t, \sin t, 2\cos t \rangle$

v) $\vec{r}(t) = \langle t\cos t, t\sin t, t \rangle$

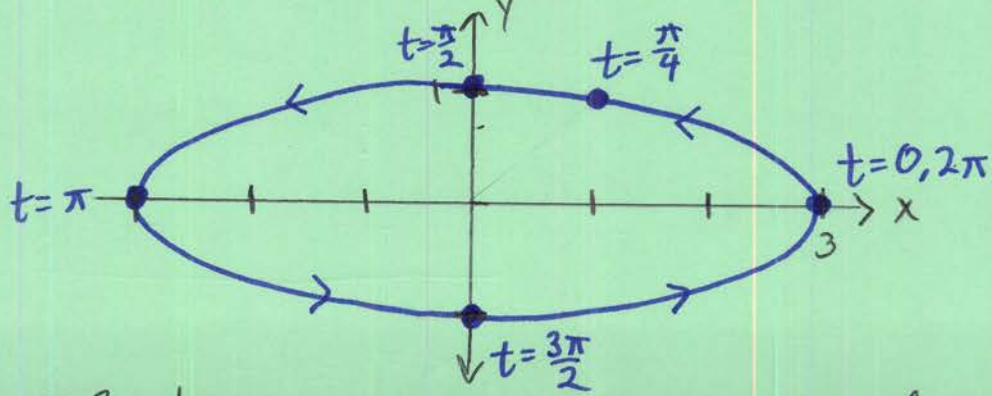
Sol:

i)



Arrows on the curve indicate direction of increasing t-values.

ii)



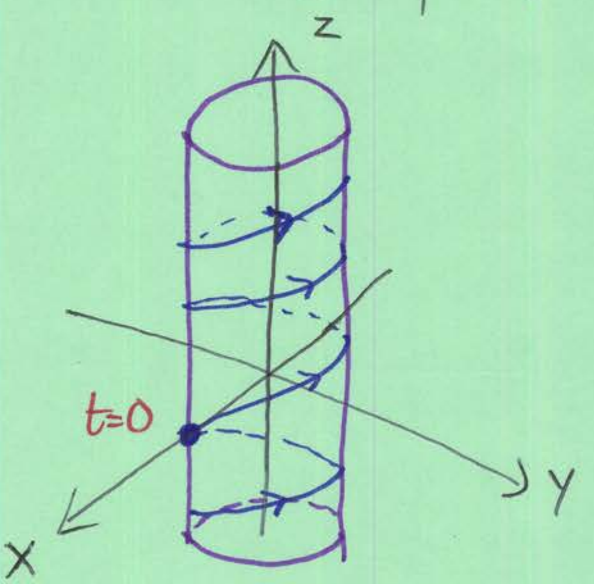
Notice that $x = 3\cos t$, $y = \sin t$ satisfies $\frac{x^2}{9} + y^2 = 1$, the eqn of an ellipse!

iii) With 3D space curves, it's often useful to find a surface that your curve sits on.

In this case we have

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle = \langle \cos t, \sin t, t\rangle$$

so $[x(t)]^2 + [y(t)]^2 = 1$. This means the curve sits on the cylinder and climbs up it as t increases:



iv) Here, $[y(t)]^2 + \frac{[z(t)]^2}{4} = 1$, so the curve sits on the elliptic cylinder $y^2 + \frac{z^2}{4} = 1$ (it opens along the x-axis).

v) This one satisfies

$$[x(t)]^2 + [y(t)]^2 = t^2(\cos^2 t + \sin^2 t) = t^2 = [z(t)]^2$$

meaning it sits on the cone $x^2 + y^2 = z^2$.

(See Mathematica code for graphs.)